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# New Exact Solutions of Steady Plane Flows of an In Compressible Fluid of Variable Viscosity

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## ABSTRACT

Some new exact solutions to the equations governing the steady plane motion of an in compressible fluid of variable viscosity for the chosen form of the vorticity distribution are determined by using transformation technique. In this case the vorticity distribution is proportional to the stream function perturbed by the product of a uniform stream and an exponential stream.

**Keywords:** Exact solutions, fluid of variable viscosity, Navier-Stokes equations, plane steady flows.

## 1. INTRODUCTION

Navier-Stoke's equations are very useful because they describe the physics of many things of academic and economic interest. Navier- Stokes equations help in the design of aircraft, cars, study of flood flows, power station designs and many more. Navier-Stokes equations are non-linear partial differential equations but in some cases this can be simplified to the linear equations, e.g. in one dimension flow and stokes flow. This non-linearity makes most of the problems impossible to solve and is the main contributor to the turbulence that the equation model.

Exact solutions are very important because they are solutions of some fundamental flows and serve as accuracy checks for experimental as well as numerical methods. Wang [5] has given an excellent review of these solutions, by prescribing vorticity function. Some of the exact solutions of Navier-

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Stokes equation are found in literature. In this case governing equations in terms of the stream function reduces to the linear equations. Taylor [1], Kampe de Feriet [2], Kovasznay [3], Lin and Tobak [4], taking viscosity as $\nabla^2\psi = K\psi$, $\nabla^2\psi = f(\psi)$, $\nabla^2\psi = y + (K^2 - 4\pi^2)\psi$,$\nabla^2\psi = A\psi + Cy$ and $\nabla^2\psi = K(\psi - R)y$, respectively used this approach. A number of exact solutions are also obtained by O.P Chandna and Oku-UK Pong [6], JI Siddique [7], A. Venkatalazmi et all [8], R.K Naeem and M. Jamil [9], S. Islam et all [11], T. Hayat et all [12], R.K Naeem and S. Younus [14], Pankaj Mishra et all [15].

In this paper we extended M. Jamil [13] work to the study flows of an incompressible fluid with variable viscosity and heat transfer, when vorticity function is given by $\nabla^2\psi = K(\psi - U(ax + by)e^{ax+by})$.

The outline of this paper is as follows. In section 2, basic flow equations are presented and transformed into a new system by defining a new variable $\xi$ by equation (22). In section 3, exact solutions of this new system are obtained. In section 4, discussion of the obtained results is presented. Conclusion is given in section 5.

## 2. BASIC EQUATIONS

The basic non-dimensional equation governing the steady plane flow of an incompressible fluid of variable viscosity, in the absence of external force and with no heat additions defined in Naeem and Jamil [10] are:

$$u_x + v_y = 0 \tag{1}$$

$$uu_x + vu_y = -p_x + \frac{1}{Re}\left[(2\mu u_x)_x + \left(\mu\left(u_y + v_x\right)\right)_y\right] \tag{2}$$

$$uv_x + vv_y = -p_y + \frac{1}{Re}\left[\left(2\mu v_y\right)_y + \left(\mu\left(u_y + v_x\right)\right)_x\right] \tag{3}$$

$$uT_x + vT_y = \frac{1}{RePr}\left(T_{xx} + T_{yy}\right) + \frac{Ec}{Re}\mu\left[2\left({u_x}^2 + {v_y}^2\right) + \left(u_y + v_x\right)^2\right] \tag{4}$$

where Reynolds number (Re), Prandtl number (Pr) are defined as

$$Re = \frac{\bar{S}\bar{U}\bar{L}}{\bar{\mu}_\circ} \tag{5}$$

$$Pr = \frac{\bar{\mu}_\circ \bar{C}_p}{\bar{K}} \tag{6}$$

$u, v$are the velocity components, $p$ is the pressure and $T$ is the temperature distribution.

Introducing the stream function $\psi(x, y)$ such that

$$u = \psi_y,\ v = -\psi_x \tag{7}$$

The system of equations (1-4), on using equation (7) transforms into the following system of equations:

$$L_x = -\psi_x w + \frac{1}{Re}\left[\mu\left(\psi_{yy} - \psi_{xx}\right)\right]_y \tag{8}$$

$$L_y = -\psi_y w + \frac{1}{Re}\left[\mu\left(\psi_{yy} - \psi_{xx}\right)\right]_x - \frac{4}{Re}\left(\mu\psi_{xy}\right)_y \tag{9}$$

$$\psi_y T_x - \psi_x T_y = \frac{Ec}{Re}\mu\left[4\left(\psi_{xy}\right)^2 + \left(\psi_{yy} - \psi_{xx}\right)^2\right] \tag{10}$$

where generalized energy function $L$ and the vorticity function $w$ are defined by

$$w = -\nabla^2\psi \tag{11}$$

$$L = p + \frac{1}{2}\left(\psi_x{}^2 + \psi_y{}^2\right) - \frac{2\mu\psi_{xy}}{Re} \tag{12}$$

Once a solution of system of equations (8-10) is obtained, the pressure $p$ can be determined from equation (12).

We will investigate fluid motion for which the vorticity distribution is proportional to the stream function perturbed by the product of a uniform stream and an exponential stream. Therefore we set

$$\psi_{xx} + \psi_{yy} = K(\psi - U(ax + by)e^{ax+by} \tag{13}$$

On substituting

$$\Psi = \psi - U(ax + by)e^{ax+by} \tag{14}$$

and using equation (13), the equation (11) becomes

$$w = -K\Psi \tag{15}$$

where $K, a, b \neq 0, a \neq b$, and $U$ are real constants.

Using equations (14) and (15), equations (8) and (9) become

$$L_x = \left(\frac{K\Psi^2}{2}\right)_x + aUK\Psi\big(1 + (ax + by)\big)e^{ax+by} +$$

$$\frac{1}{Re}\left[\mu(\Psi_{yy} - \Psi_{xx} + (b^2 - a^2)\ U(2 + (ax + by))e^{ax+by}\right]_y \tag{16}$$

$$L_y = (\frac{K\Psi^2}{2})_y + bUK\Psi\big(1 + (ax + by)\big)e^{ax+by} +$$

$$\frac{1}{Re}\left[\mu\big(\Psi_{yy} - \Psi_{xx} + (b^2 - a^2)U\big(2 + (ax + by)\big)e^{ax+by}\big)\right]_x$$

$$-\frac{4}{Re}[\ \mu(\Psi_{xy} + abU(2 + (ax + by))e^{ax+by})]_y \tag{17}$$

Using integrability condition $L_{xy} = L_{yx}$, equation (16) and (17) yields

$$J_{xx} - J_{yy} + UK\big(b\Psi_x - a\Psi_y\big)[1 + (ax + by)]e^{ax+by} - \frac{4}{Re}\big[\mu\big(\Psi_{xy} + abU\big(2 + (ax +$$

$$byeax+byxy=0 \tag{18}$$

where $J$ is defined as

$$J = \frac{1}{Re}[\mu(\Psi_{yy} - \Psi_{xx} + U(b^2 - a^2)(2 + (ax + by)e^{ax+by})] \tag{19}$$

Equation (18) must be satisfied by the viscosity $\mu$ and the function $\Psi$ for the motion of a steady incompressible distribution, defined by equation (14).

Equation (10), using equation (14) becomes

$$(\Psi_y + bU[1 + (ax + by)]e^{ax+by}T_x - (\Psi_x + aU[1 + (ax + by)]e^{ax+by})T_y = \frac{1}{RePr}\big(T_{xx} + T_{yy}\big) +$$

$$EcRe\mu[4\Psi xy+abU2+ax+byeax+by2+\Psi yy-\Psi xx+Ub2-a22+ax+byeax+by2]$$

(20)

Equation (13), using equation (14), becomes

$$\Psi_{xx} + \Psi_{yy} - K\Psi = -(a^2 + b^2)[2 + (ax + by)]e^{ax+by} \qquad (21)$$

On introducing the new variable '$\xi$' defined by

$$\xi = ax + by$$

The equations (18), (20), and (21) become

$$\Psi_{\xi\xi} - A\Psi = -U(2 + \xi)e^{\xi} \qquad (22)$$

$$T_{\xi\xi} + EcPrA^2(a^2 + b^2)\mu\Psi^2 = 0 \qquad (23)$$

$$(\mu\Psi)_{\xi\xi} = 0 \qquad (24)$$

where, $A = \frac{K}{a^2+b^2}$

## 3. EXACT SOLUTIONS

In this section we determined the solutions of the system of equations (22-24) for the following three cases:

Case I: $A = -n^2, \quad n > 0$

Case II: $A = m^2, m > 0$

Case III: $A = 0$

Let us now determine the solution to the equations (22-24) and (14) for the chosen vorticity distribution that will further use to determine $\mu, T, \psi, u, v$ and $p$ from the system of equations (23), (24), (14), (7) and (12), taking these three cases separately.

**Case I:**

For this case solution of the equation (22) in the physical plane is given by

$$\Psi = l_1 \cos(n(ax+by)+l_2) - \frac{U}{(1+n^2)^2}(2n^2+(1+n^2)(ax+by))e^{ax+by} \tag{25}$$

where$l_1$ $and$ $l_2$ are real constants.

Equation (24), employing equation (25) yields

$$\mu = \frac{l_3(ax+by)+l_4}{l_1\cos(n(ax+by)+l_2) - \frac{U}{(1+n^2)^2}(2n^2+(1+n^2)(ax+by)).e^{ax+by}} \tag{26}$$

where $l_3, l_4$ $are\ real\ constants.$ Equation (23) utilizing equation (25), becomes

$$T_{\xi\xi} + EcPrA^2(a^2+b^2)(l_3\xi+l_4)\Psi = 0 \tag{27}$$

The solution of equation (27) is

$$T = \frac{EcPrn(a^2+b^2)}{1+n^2}\Big\{l_3\Big[Un^3\Big((\xi-2)^2e^{\xi} - \frac{Un}{1+n^2}(\xi-2)e^{\xi} + (1+n^2)l_1(n\xi\cos(n\xi+l_2) -$$

$$2\sin(n\xi+l2$$

$$+l_4n[Un^2(\xi-1)e^{\xi} + \frac{2U}{1+n^2}e^{\xi} + (1+n^2)l_1\cos(n\xi+l_2)]\} + l_5\xi + l_6 \tag{28}$$

where $l_5, l_6$ are real constants.

In this case the stream function $\psi$, velocity components and pressure distribution are given by

$$\psi = \frac{Un^2}{1+n^2}(ax+by)e^{ax+by} - \frac{2Un^2}{(1+n^2)^2}e^{ax+by} + l_1\cos(n(ax+by)+l_2) \tag{29}$$

$$u = \frac{Un^2b}{1+n^2}[1+(ax+by)]e^{ax+by} - l_1nb\sin(n(ax+by)+l_2) - \frac{2Ubn^2}{(1+n^2)^2}e^{ax+by} \tag{30}$$

$$v = -\frac{Un^2a}{1+n^2}[1+(ax+by)]e^{ax+by} + l_1na\sin(n(ax+by)+l_2) + \frac{2Uan^2}{(1+n^2)^2}e^{ax+by} \tag{31}$$

$$p = \int[\psi_x(\nabla^2\psi)]dx - \frac{l_3n^2(b^2-a^2)}{Re}bx - \frac{2abn^2}{Re}[l_3(ax+by)+l_4] -$$

$$\frac{n^2(a^2+b^2)}{2(n^2+1)^2}[U(ax+by)e^{ax+by}+\frac{n^2-1}{n^2+1}Ue^{ax+by}-\frac{l_1}{n}\sin[n(ax+by)+l_2]^2+l_7 \qquad (32)$$

where $l_7$ is a real constant.

**Case II:**

For this case the solution of the equation (22) is

$$\Psi = M_1e^{m(ax+by)}+M_2e^{-m(ax+by)}+\frac{U}{m^2-1}(ax+by)e^{ax+by}+\frac{2Um^2}{(m^2-1)^2}e^{ax+by} \qquad (33)$$

where $M_1$ , $M_2$are real constants, equation (24) using equation (33) gives

$$\mu = \frac{M_3\xi+M_4}{M_1e^{m(ax+by)}+M_2e^{-m(ax+by)}+\frac{U}{m^2-1}(ax+by)e^{ax+by}+\frac{2Um^2e^{ax+by}}{(m^2-1)^2}} \qquad (34)$$

where$M_3$ and $M_4$ are real constants.

Solution of the equation (23), using equation (33) is

$$T_{\xi\xi}+EcPrm^2(a^2+b^2)(M_3\xi+M_4)\Psi = 0 \qquad (35)$$

Solution to the equation (35) is

$$T = \frac{EcPrm(a^2+b^2)}{1-m^2}\left[\begin{matrix} M_3\left\{\begin{matrix}\frac{Ue^{\xi}m^3}{m^2-1}(\xi-2)[(m^2-1)\xi+2]+M_1(m^2-1)(m\xi-2)e^{m\xi}+ \\ M_2(m^2-1)(m\xi+2)e^{-m\xi}\end{matrix}\right\}+ \\ mM_4\left\{\frac{Ue^{\xi}m^2}{m^2-1}(\xi+1)(m^2+1)+M_1(m^2-1)e^{m\xi}+M_2(m^2-1)e^{-m\xi}\right\}\end{matrix}\right]+$$

$$M_5\xi+M_6 \qquad (36)$$

where $M_5$ and $M_6$ are real constants.

In this case the stream function $\psi$, components of the velocity distribution and pressure are given by

$$\psi = \frac{Um^2}{m^2-1}(ax+by)e^{ax+by} + M_1 e^{m(ax+by)} + M_2 e^{-m(ax+by)} + \frac{2Um^2}{(m^2-1)^2} e^{ax+by} \tag{37}$$

$$u = \frac{Ubm^2}{m^2-1}[1+(ax+by)]e^{ax+by} + bmM_1 e^{m(ax+by)} - bmM_2 e^{-m(ax+by)} + \frac{2Ubm^2}{(m^2-1)^2} e^{ax+by} \tag{38}$$

$$v = -\frac{Uam^2}{m^2-1}[1+(ax+by)]e^{ax+by} - amM_1 e^{m(ax+by)} + amM_2 e^{-m(ax+by)} - \frac{2Uam^2}{(m^2-1)^2} e^{ax+by} \tag{39}$$

$$p = \int [\psi_x(\nabla^2\psi)]dx + \frac{M_3 m^2(b^2+a^2)}{Re} bx + \frac{2abm^2}{Re}[M_3(ax+by)+M_4] -$$

$$\frac{(a^2+b^2)}{2(m^2-1)^2}[Um^2(ax+by)e^{ax+by} + mM_1(m^2-1)e^{m(ax+by)} - mM_2(m^2-1)e^{-m(ax+by)}]^2 + M_7 \tag{40}$$

where $M_7$ is a real constant

**Case III:**

For this case, we have

$$\Psi = N_1(ax+by) + N_2 - U(ax+by)e^{ax+by} \tag{41}$$

$$u = \frac{N_{3(ax+by)} + N_4}{N_1(ax+by) + N_2 - U(ax+by)e^{ax+by}} \tag{42}$$

$$T = N_5(ax+by) + N_6 \tag{43}$$

where $N_1$, $N_2$, $N_3$,……$N_6$ are real constants.

A=0, corresponds to an irrational flow and it is following uniform flow

$$\psi = N_1(ax+by) + N_2 \tag{44}$$

The velocity components and pressure distribution is given by

$$u = N_1 b \tag{45}$$

$$v = -N_1 a \quad (46)$$

$$p = -\frac{(a^2+b^2)N_1}{2} + N_7 \quad (47)$$

where $N_7$ *is* a real constant.

## 4. RESULTS AND DISCUSSION

A class of new exact solutions of the steady flows of incompressible fluid of variable viscosity are obtained for which the vorticity distribution is defined by the equation (13) and different effects of the parameters *n, m, a, b* and U on the components of velocity profiles *v(x, y)* and *u(x, y)* are depicted in figures (1-8). Figures (1-4) and figures (5-8) are of case I and case II respectively. Figures (1) & (2) present the effect of parameter *n* on the velocity components *u* and *v* in the direction of *y* and *x* respectively. It is clear from these figures that velocity components *u* and *v* have oscillating behavior and both are increasing functions of *n* in absolute value. Similar effects are observed in the direction of *y* and *x* for the parameter *b* and *a* on the velocity components *v(x, y)* and *u(x, y)* respectively as shown is figures (3) and (4). Figures (5-8) show the influence of parameters *m*, *a* and *b* on the velocity components *v(x, y)* and *u(x, y)*, in the direction of *y* and *x* respectively. These figures also show that velocity components are increasing functions of parameters.

## 5. CONCLUSION

A class of new exact solutions of the equation governing the steady plane motion of incompressible fluid of variable viscosity is obtained for which the vorticity distribution is given by equation (13). The effects of parameters n, m, b, a and U of interest on the velocity components are plotted and discussed.

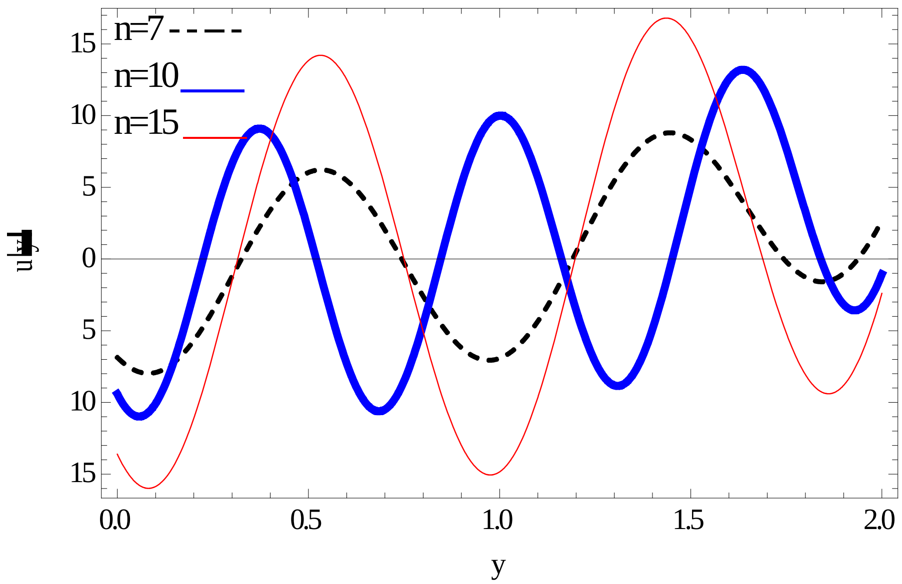


**Figure 1:** The effect of *n* on velocity component *u* for $\boldsymbol{l_1 = l_2 = U = b = 1, a = 0}$ , in the direction of y.

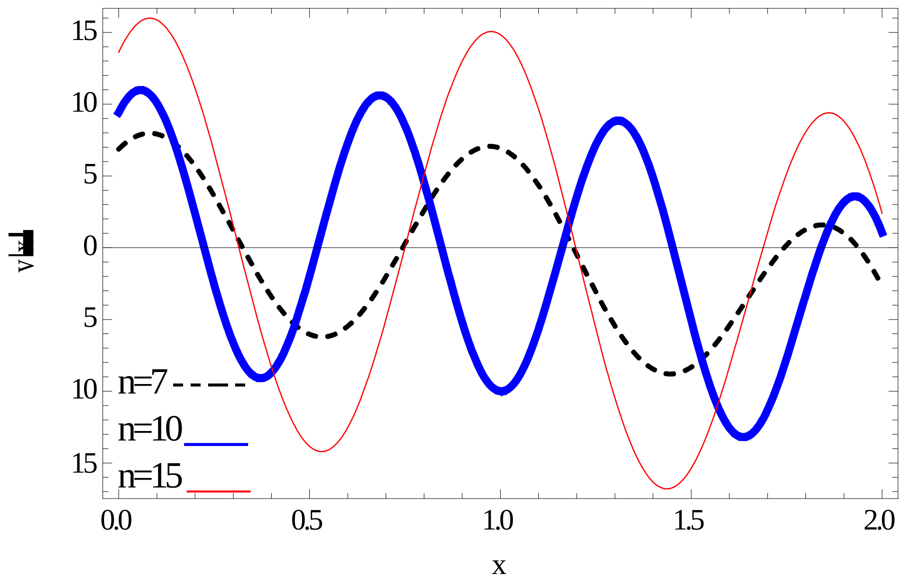


**Figure 2:** The effect of *n* on velocity component *v* for $\boldsymbol{l_1 = l_2 = U = a = 1, b = 0}$ , in the direction of *x*.

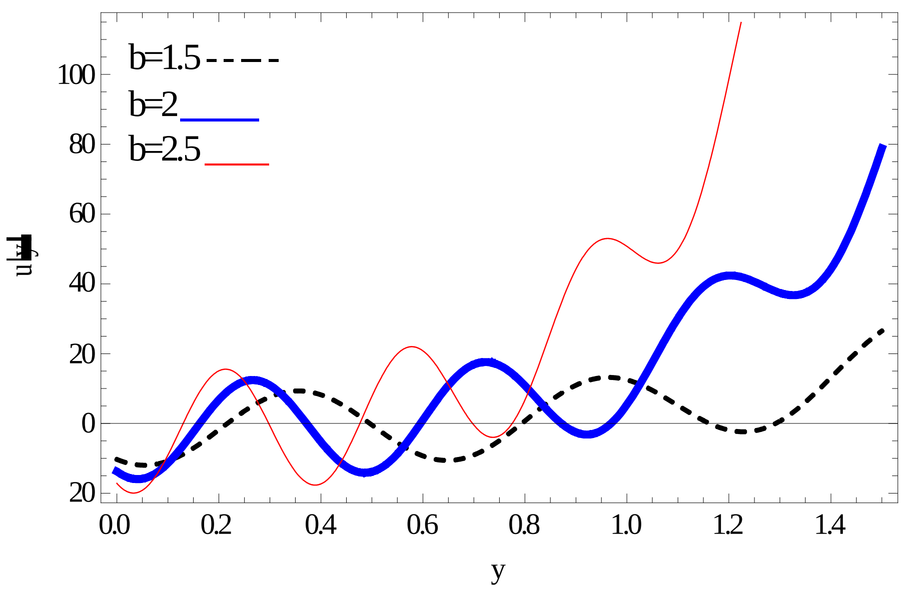


**Figure 3:** The effect of *b* on velocity component *u* for $\boldsymbol{l_1 = l_2 = U = 1, a = 0, n = 7}$ in the direction of y.

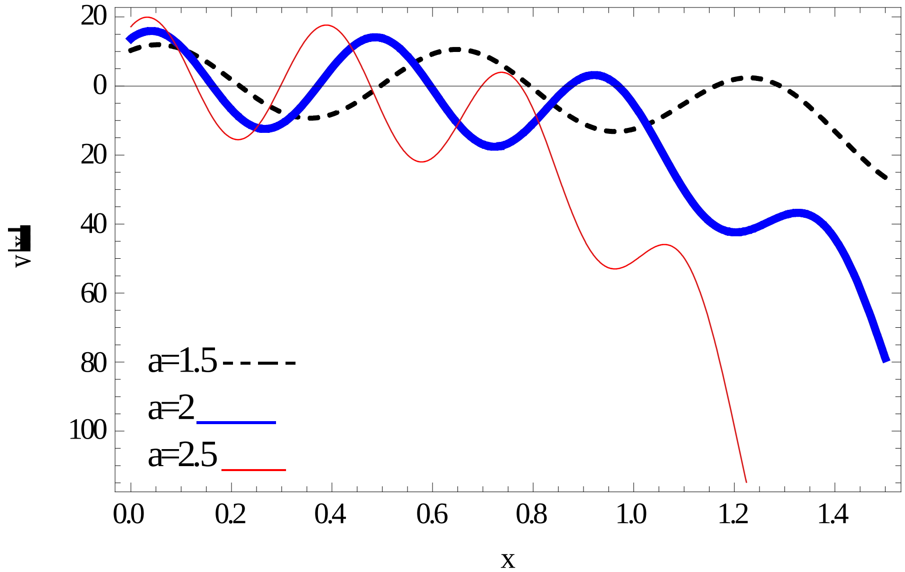


**Figure 4:** The effect of *a* on velocity component v for $\boldsymbol{l_1 = l_2 = U = 1, b = 0, n = 7}$ in the direction of x.

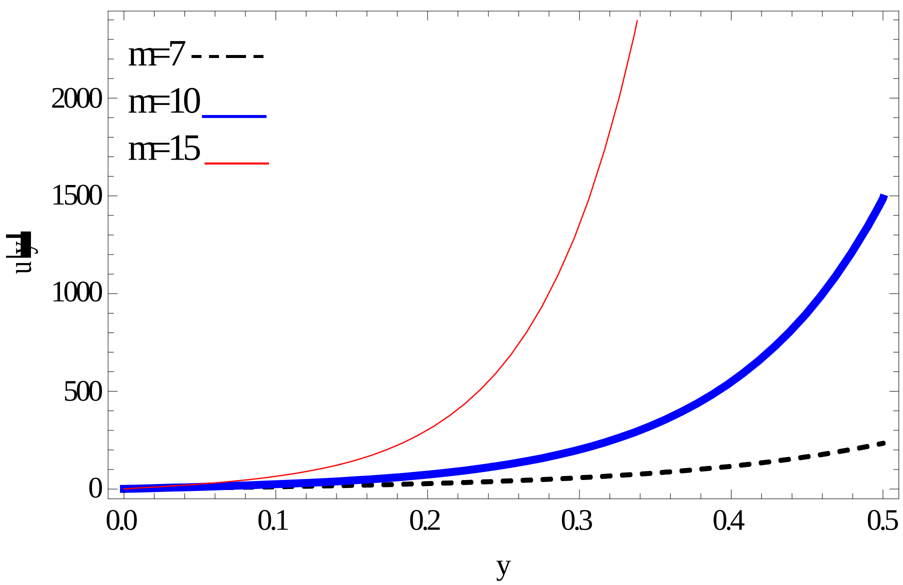


**Figure 5:** The effect of *m* on velocity component *u* for $\boldsymbol{M_1 = M_2 = U = b = 1, a = 0}$ , in the direction of y.

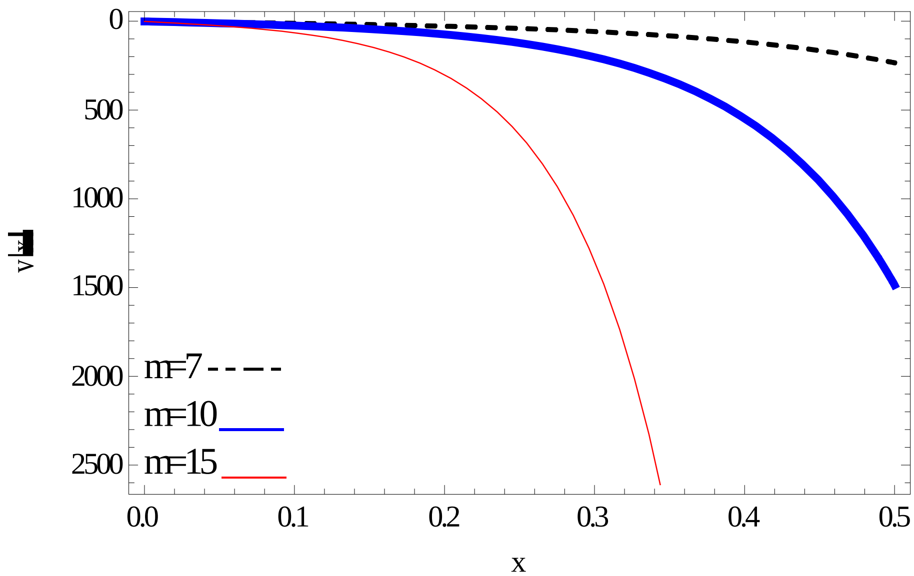


**Figure 6:** The effect of *m* on velocity component *v* for $\boldsymbol{M_1 = M_2 = U = a = 1, b = 0}$ , in the direction of *x*.

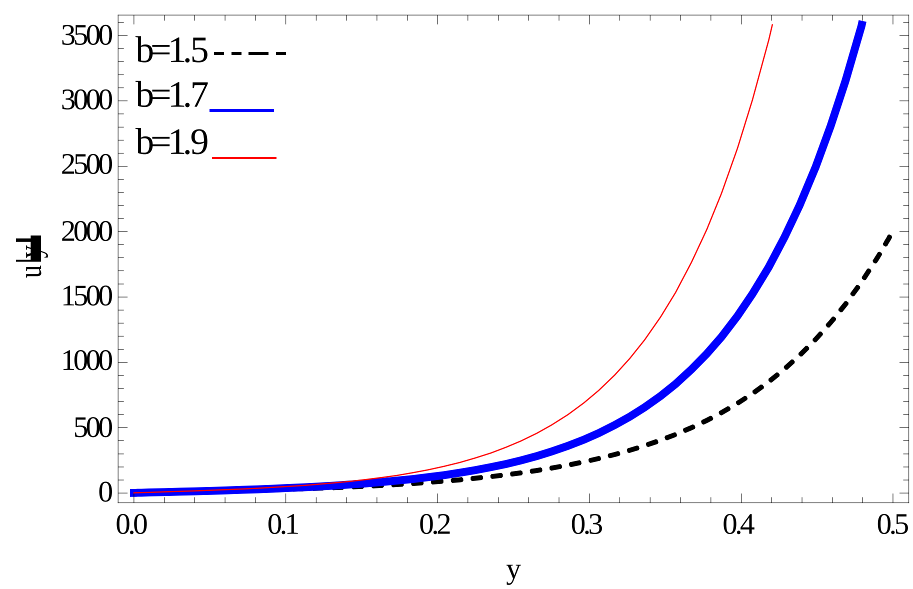


**Figure 7:** The effect of *b* on velocity component *u* for $\boldsymbol{M_1 = M_2 = U = 1, a = 0, m = 7}$  in the direction of y.

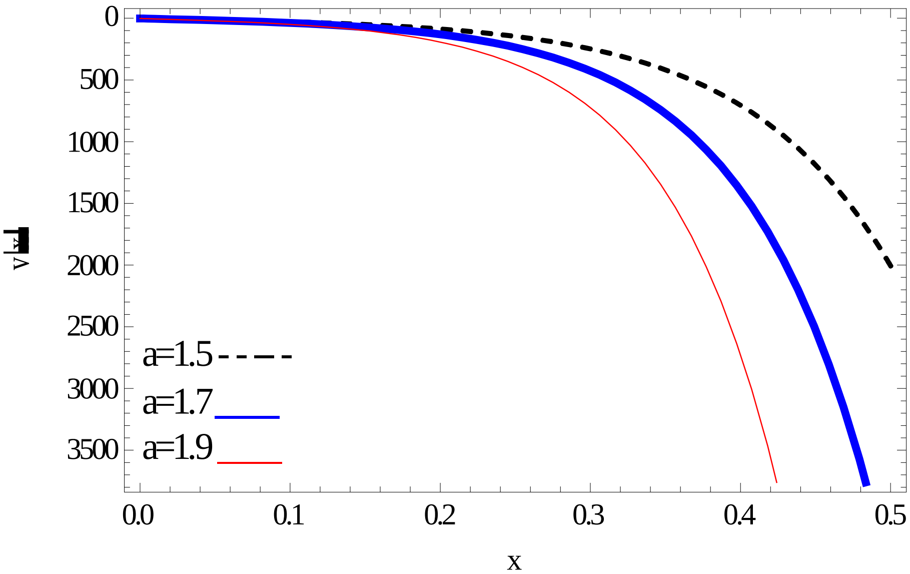


**Figure8:** The effect of *a* on velocity component *v* for $\boldsymbol{M_1 = M_2 = U = 1, b = 0, m = 7}$ in the direction of *x*.

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