

***“Some Exact Solutions of Second Grade Fluid over the  
Plane moving with Constant Acceleration”***

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**1 Introduction:**

Many materials such as clay coatings, drilling muds, suspensions, certain oils and greases, polymer melts, elastomers and many emulsions have been treated as non-Newtonian fluids. It is difficult to suggest a single model which exhibits all properties of non-Newtonian fluids as in case of the Newtonian fluids. They cannot be described in a simple model as for the Newtonian fluids and there has been much confusion over the classification of non-Newtonian fluids. However, non-Newtonian fluids may be classified as: **(i)** fluids for which the shear stress depends on the shear rate; **(ii)** fluids for which the relation between the shear stress and shear rate depends on time; **(iii)** fluids which possess both elastic and viscous properties called visco elastic fluids or elastico-viscous fluids. Because of great diversity in the physical structure of non-Newtonian fluids, it does not seem possible to recommend a single constitutive equation for use in the cases described in **(i)**, **(ii)** and **(iii)**. Therefore, many constitutive equations for non-Newtonian fluids have been proposed. Most of them are empirical or semi-empirical. For more general three-dimensional representations the method of continuum mechanics is needed. Although many constitutive equations have been suggested, many questions are still unsolved. Some of the continuum models do not give satisfactory results in accordance

with the available experimental data. Therefore, in many practical applications, empirical or semi-empirical equations have been used.

A constitutive equation is a relation between stress and the local properties of the fluid. For a fluid at rest the stress is determined wholly by the static pressure. Although in the case of a fluid in relative motion the relation between stress and the local properties of the fluid is more complicated, some modifications may be made such as the stress being dependent only on the instantaneous distribution of fluid velocity in the neighborhood of the element. This distribution may be expressed only in terms of the velocity gradient components such as for a Newtonian fluid. However, non-Newtonian fluids cannot be described as simple as Newtonian fluids. One of the most popular models for non-Newtonian fluids is the model that is called the second-order fluid or second grade fluid [6].

## **2 Literature Review:**

Although there are some criticisms on the applications of this model [5, 9, 10], many papers have been published and a listing of some of them may be found in the literature. The constitutive equation of a second grade fluid is a linear relation between the stress and the first Rivlin-Ericksen tensor and the square of the first Rivlin-Ericksen tensor and the second Rivlin-Ericksen tensor [6]. The constitutive equation has three coefficients. There are some restrictions on these coefficients due to the Clausius-Duhem inequality and due to assumption that the Helmholtz free energy is minimum in equilibrium. A comprehensive discussion on the restrictions for these coefficients has been given by Dunn and Fosdick [3], and Dunn and Rajagopal [2]. One of these coefficients describes the viscosity coefficient similar to Newtonian fluids. The restrictions on the other two coefficients have not been confirmed by experiments and the sign of the material moduli is the subject of much controversy [4, 7]. The conclusion is that the fluids which have been tested are not second grade fluids and they are characterized by a different constitutive structure.

The equation of motion of incompressible second grade fluids is of higher order than the Navier–Stokes equation. The Navier–Stokes equation is a second order partial differential equation, but the equation of motion of a second grade fluid is a third order partial differential equation. A marked difference between the case of the Navier–Stokes theory and that for fluids of second grade is that ignoring the non-linearity in Navier–Stokes does not lower the order of the equation, however, ignoring the higher order non-linearities in the case of the second grade fluids, reduces the order of the equation.

The no-slip boundary condition is sufficient for a Newtonian fluid, but may not be sufficient for a fluid of second grade, based on the previous experience with the partial differential equation. Therefore, one needs an additional condition at boundary. In the case of initial boundary value problem, the no-slip boundary condition suffices. A critical review on the boundary conditions, the existence and uniqueness of the solutions has been given by Rajagopal [8]. In order to overcome the difficulty, several workers have studied acceptable additional conditions.

### 3 Basic Governing Equations:

The Cauchy stress  $\mathbf{T}$  in an incompressible homogeneous fluid of second grade is given as [1]

$$\begin{aligned} \mathbf{T} &= -\rho\mathbf{I} + \mathbf{S}, \\ \mathbf{S} &= \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \end{aligned} \quad (3.1)$$

where  $-\rho\mathbf{I}$  is the indeterminate part of the stress due to the constraint of incompressibility,  $\mathbf{S}$  is the extra-stress tensor,  $\mu$  is the dynamic viscosity,  $\alpha_1$  and  $\alpha_2$  are the normal stress moduli and  $\mathbf{A}_1$  and  $\mathbf{A}_2$  are the kinematic tensors [6] defined by (1.5) and (1.6). Since the fluid is incompressible, it can undergo only isochoric motions and hence

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0. \quad (3.2)$$

For the problem under consideration, we shall assume a velocity field of the form:

$$\mathbf{V} = V(y, t) = u(y, t)\mathbf{i}, \quad \mathbf{S} = S(y, t), \quad (3.3)$$

consider here, where  $\mathbf{i}$  is the unit vector along the x-direction of the Cartesian coordinates system. The constraint of incompressibility (4.2) is automatically satisfied for these flows. If the fluid is at rest up to the moment  $t = 0$ , then

$$\mathbf{V} = V(y, 0) = 0, \quad \mathbf{S} = S(y, 0) = 0, \quad (3.4)$$

The governing equations corresponding to such motion are:

$$\frac{\partial u}{\partial t} = \left( \nu + \alpha \frac{\partial}{\partial t} \right) \frac{\partial^2 u}{\partial t^2}, \quad (3.5)$$

$$\tau = \left( \mu + \alpha_1 \frac{\partial u}{\partial y} \right) \frac{\partial u}{\partial t}, \quad (3.6)$$

where,  $\nu = \frac{\mu}{\rho}$  is the kinematic viscosity and  $\alpha = \frac{\alpha_1}{\rho}$ .

Consider an incompressible second grade fluid occupying the space above a flat plate perpendicular to y-axis. Initially, the fluid is at rest and at the moment  $t = 0^+$  the plate is brought to the velocity  $U\mathbf{i}$  in its plane. Due to the shear, the fluid above the plate is gradually moved, while the governing equations are given by (3.5) and (3.6). The relevant problem under initial and boundary conditions is:

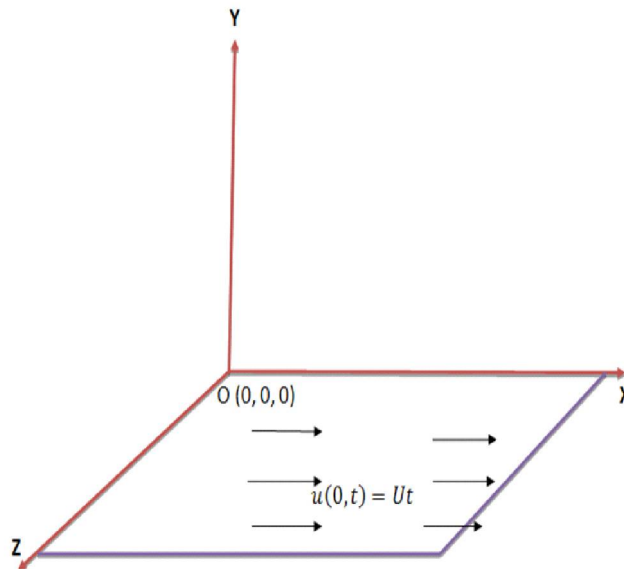
$$u(y, 0) = \frac{\partial u(y, 0)}{\partial t} = 0, \quad y > 0, \quad (3.7)$$

$$u(0, t) = Ut, \quad t \geq 0. \quad (3.8)$$

Moreover, the natural conditions

$$u(y, t), \frac{\partial u(y, t)}{\partial y} \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty, \quad t > 0, \quad (3.9)$$

have to be also satisfied. They are consequences of the fact that the fluid is at rest at infinity and there is no shear in the free stream.



**Fig. 4.1:** Geometry of the problem of **second grade fluid** over a plane moving with the velocity  $u(0, t) = Ut$ .

The solution of the problem lies in the calculation of velocity field i.e.  $u(y, t)$  and the adequate shear stress  $\tau(y, t)$  associated with the velocity obtained. Hence, for the exact solution, we are going to first calculate the velocity field and then with the help of obtained velocity result, corresponding shear stress can be calculated.

In order to determine the exact solution, we shall use **(5)** the Fourier sine transform from the list of useful formulae. Multiplying both sides of **(3.5)** by,  $\sqrt{\frac{2}{\pi}} \sin(y\xi)$ , integrating the results with respect to  $y$  from 0 to infinity, and taking into account the boundary condition **(3.8)**, we find that:

$$\frac{\partial u_s}{\partial t} = \left( \nu + \alpha \frac{\partial}{\partial t} \right) \left[ -\xi^2 u_s + \sqrt{\frac{2}{\pi}} \xi U t \right] \quad (4.1)$$

where, the Fourier sine transform  $u_s = u_s(\xi, t)$  of  $u(y, t)$  is defined as:

$$u_s(\xi, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(y, t) \sin(y\xi) dy,$$

has to satisfy the initial conditions

$$u_s(\xi, 0) = \frac{\partial u_s(\xi, 0)}{\partial t} = 0, \quad \xi > 0. \quad (4.2)$$

Now, eq. **(4.1)** can be written as:

$$\frac{\partial u_s}{\partial t} + \xi^2 \left( \nu + \alpha \frac{\partial}{\partial t} \right) u_s = \nu \xi U \sqrt{\frac{2}{\pi}} t + \alpha \xi U \sqrt{\frac{2}{\pi}} (1).$$

By applying the Laplace transform to **(4.1)** and having in mind the initial conditions **(4.2)**, we find that:

$$[(1 + \alpha \xi^2)q + \nu \xi^2] \bar{u}_s = \nu \xi U \sqrt{\frac{2}{\pi}} \frac{1}{q^2} + \alpha \xi U \sqrt{\frac{2}{\pi}} \frac{1}{q}.$$

It can also be written as:

$$\bar{u}_s = U \sqrt{\frac{2}{\pi}} \xi \frac{1}{q^2} \left[ \frac{v + \alpha q}{(1 + \alpha \xi^2)q + v\xi^2} \right],$$

$$\bar{u}_s = U \sqrt{\frac{2}{\pi}} \frac{1}{\xi} \left[ \frac{1}{q^2} - \frac{1}{q[(1 + \alpha \xi^2)q + v\xi^2]} \right]. \quad (4.3)$$

Now, for a more suitable presentation of the final results, we rewrite (4.3) in the following equivalent form:

$$\bar{u}_s = U \sqrt{\frac{2}{\pi}} \left[ \frac{1}{\xi} \frac{1}{q^2} - \frac{1}{v\xi^3} \frac{1}{q} + \frac{1}{v\xi^3} \frac{\alpha \xi^2 + 1}{(1 + \alpha \xi^2)q + v\xi^2} \right]. \quad (4.4)$$

Inverting (4.4) by means of the Fourier sine transform, we can write  $\bar{u}$  as:

$$\bar{u} = \frac{2U}{\pi} \int_0^\infty \frac{\sin(y\xi)}{\xi} \left[ \frac{1}{q^2} - \frac{1}{v\xi^2} \frac{1}{q} + \frac{1}{v\xi^2} \frac{\alpha \xi^2 + 1}{(1 + \alpha \xi^2)q + v\xi^2} \right] d\xi. \quad (4.5)$$

Finally, in order to obtain the velocity field  $u(y, t) = \mathcal{L}^{-1}\{\bar{u}(y, q)\}$ , we apply the inverse Laplace transform to (4.5). As a result, we find for the velocity field, the following simple expression:

$$u = Ut - \frac{2U}{\pi} \int_0^\infty \frac{\sin(y\xi)}{v\xi^3} \left[ 1 - \exp\left(\frac{-v\xi^2 t}{1 + \alpha \xi^2}\right) \right] d\xi.$$

More simply, we can write the final equation for the velocity as:

$$u = Ut - \frac{2U}{v\pi} \int_0^\infty \frac{\sin(y\xi)}{\xi^3} \left[ 1 - \exp\left(\frac{-v\xi^2 t}{1 + \alpha\xi^2}\right) \right] d\xi. \quad (4.6)$$

The above obtained velocity expression can also be written as:

$$u(y, t) = u_{LS}(y, t) + u_{TS}(y, t), \quad (4.7)$$

where,

$$u_{LS}(y, t) = Ut - \frac{2U}{v\pi} \int_0^\infty \frac{\sin(y\xi)}{\xi^3} d\xi, \quad (4.8)$$

is the large time solution of the velocity field and

$$u_{TS}(y, t) = \frac{2U}{v\pi} \int_0^\infty \frac{\sin(y\xi)}{\xi^3} \exp\left(\frac{-v\xi^2 t}{1 + \alpha\xi^2}\right) d\xi, \quad (4.9)$$

is the transient part.

Partially differentiating the velocity obtained in (4.6) with respect to "y", we get the following result:

$$\frac{\partial u}{\partial y} = -\frac{2U}{v\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2} \left[ 1 - \exp\left(\frac{-v\xi^2 t}{1 + \alpha\xi^2}\right) \right] d\xi. \quad (4.10)$$

Now, partially differentiating (4.10) with respect to "t", we get:

$$\frac{\partial}{\partial t} \left( \frac{\partial u}{\partial y} \right) = -\frac{2U}{v\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2} \left( \frac{v\xi^2}{1 + \alpha\xi^2} \right) \exp\left(\frac{-v\xi^2 t}{1 + \alpha\xi^2}\right) d\xi. \quad (4.11)$$



Further, substituting the results obtained in (4.10) and (4.11) into (3.6), we get the following result:

$$\tau = -\frac{2\mu U}{v\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2} \left[ 1 - \exp\left(\frac{-v\xi^2 t}{1 + \alpha\xi^2}\right) \right] d\xi$$

$$\frac{2U\alpha_1}{v\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2} \left( \frac{v\xi^2}{1 + \alpha\xi^2} \right) \exp\left(\frac{-v\xi^2 t}{1 + \alpha\xi^2}\right) d\xi.$$

After simplifications, the above obtained result can be written as:

$$\tau = -\frac{2\mu U}{v\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2} d\xi$$

$$+ \frac{2U}{v\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2} \left[ \mu - \frac{\alpha_1 v \xi^2}{1 + \alpha \xi^2} \right] \exp\left(\frac{-v\xi^2 t}{1 + \alpha \xi^2}\right) d\xi,$$

$$\tau = -\frac{2\rho U}{\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2} d\xi$$

$$+ \frac{2\rho U}{\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2(1 + \alpha\xi^2)} \exp\left(\frac{-v\xi^2 t}{1 + \alpha\xi^2}\right) d\xi.$$

Finally, the most simplified expression for the shear stress is:

$$\tau = -\frac{2\rho U}{\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2} \left[ 1 - \frac{1}{(1 + \alpha\xi^2)} \exp\left(\frac{-v\xi^2 t}{1 + \alpha\xi^2}\right) \right] d\xi. \quad (4.12)$$

The above obtained expression for the shear stress can also be written as:

$$(y, t) = \tau_{LS}(y) + \tau_{TS}(y, t), \quad (4.13)$$

where,

$$\tau_{LS}(y, t) = -\frac{2\rho U}{\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2} d\xi, \quad (4.14)$$

and

$$\tau_{TS}(y, t) = \frac{2\rho U}{\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2(1 + \alpha\xi^2)} \exp\left(\frac{-v\xi^2 t}{1 + \alpha\xi^2}\right) d\xi, \quad (4.15)$$

are large time and transient parts respectively.

## 5 Special Case: Newtonian Fluid:

Making  $\alpha_1 = 0$  or  $\alpha = 0$  in (4.6) and (4.12), we obtain the velocity field and associated shear stress corresponding to the Newtonian fluid performing the same motion. We get the velocity field as:

$$u_N(y, t) = Ut - \frac{2U}{v\pi} \int_0^\infty \frac{\sin(y\xi)}{\xi^3} [1 - \exp(-v\xi^2 t)] d\xi, \quad (4.16)$$

and the shear stress

$$(y, t) = -\frac{2\mu\rho U}{\pi} \int_0^\infty \frac{\cos(y\xi)}{\xi^2} [1 - \exp(-v\xi^2 t)] d\xi. \quad (4.17)$$

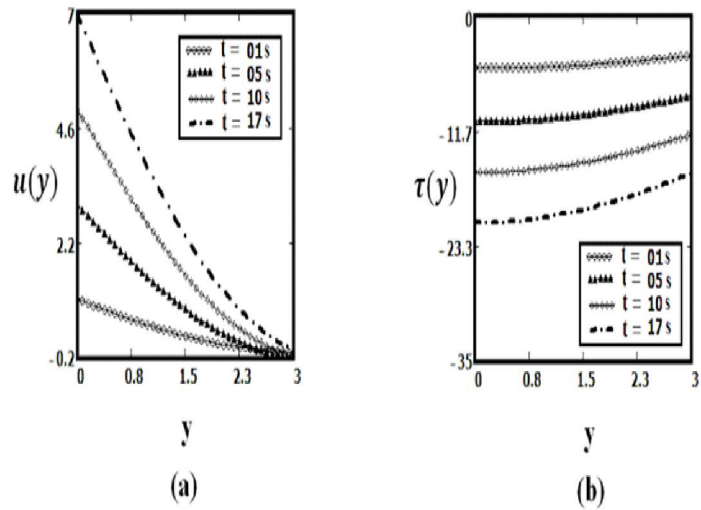


Fig. 4.2: Profiles of the velocity field  $u(y,t)$  and the shear stress  $\tau(y,t)$  given by Eqs. (4.16) and (4.22) for  $U = 1, \alpha = 0.2, \rho = 45.385, \nu = 0.2288$  and different values of  $t$ .

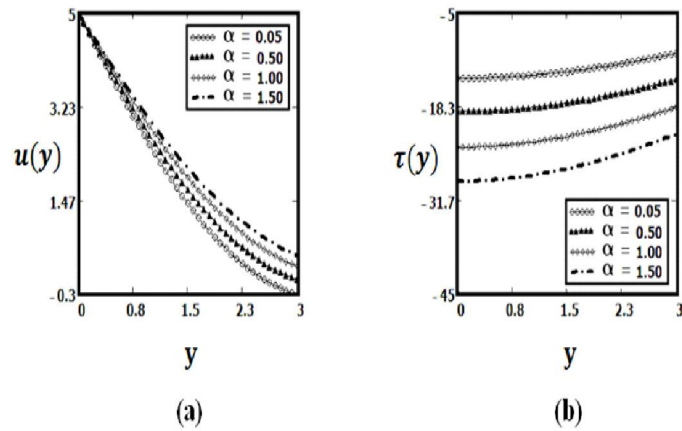


Fig. 4.3: Profiles of the velocity field  $u(y,t)$  and the shear stress  $\tau(y,t)$  given by Eqs. (4.16) and (4.22) for  $U = 1, \rho = 45.385, \nu = 0.2288, t = 5\text{ s}$  and different values of  $\alpha$ .

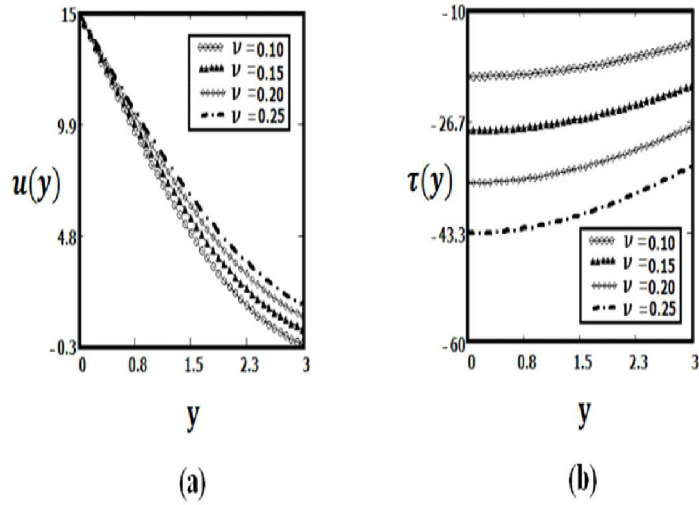


Fig. 4.4: Profiles of the velocity field  $u(y,t)$  and the shear stress  $\tau(y,t)$  given by Eqs. (4.16) and (4.22) for  $U = 1, \alpha = 0.2, \rho = 45.385, t = 15s$  and different values of  $\nu$ .

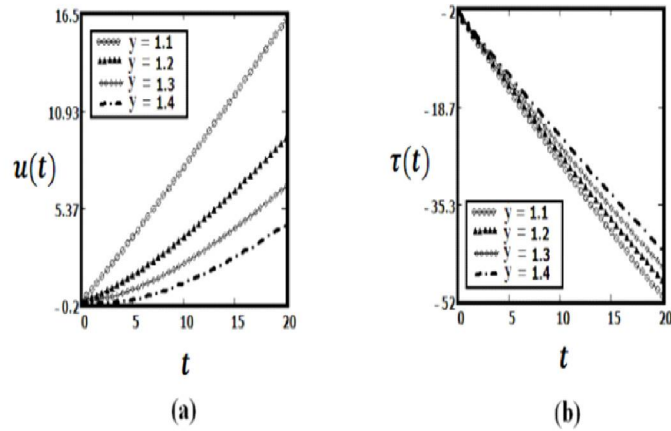
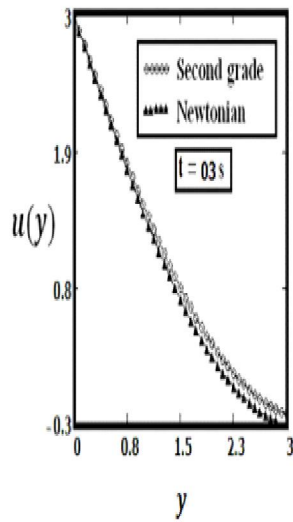
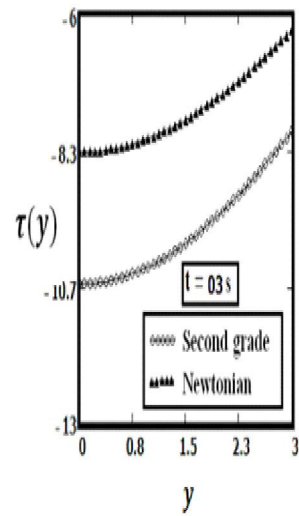


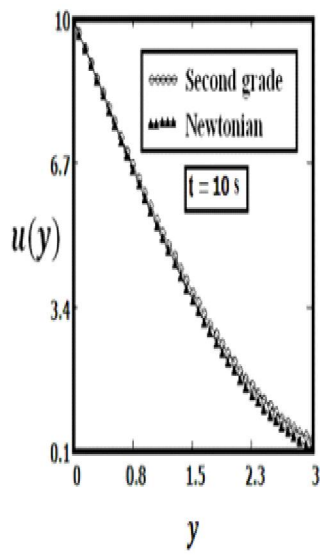
Fig. 4.5: Profiles of the velocity field  $u(y,t)$  and the shear stress  $\tau(y,t)$  given by Eqs. (4.16) and (4.22) for  $U = 1, \alpha = 0.2, \rho = 45.385, \nu = 0.2288$  and different values of  $y$ .



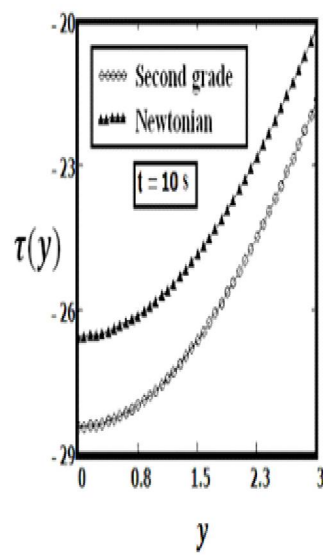
(a1)



(b1)



(a2)



(b2)

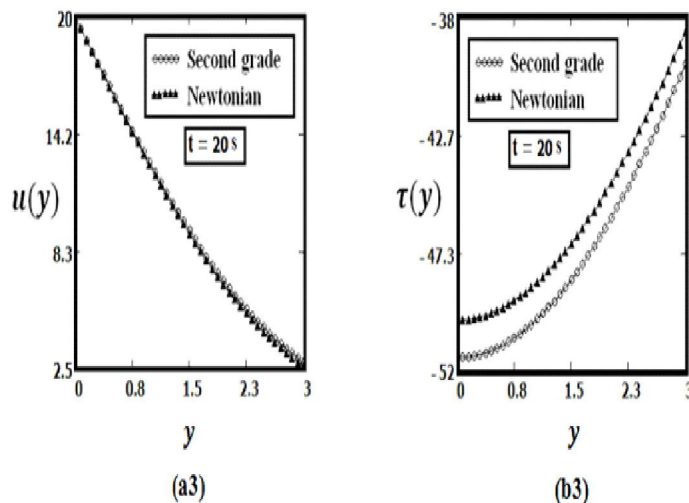


Fig. 4.6: Profiles of the velocity field  $u(y, t)$  and the shear stress  $\tau(y, t)$  given by Eqs. (4.16), (4.25), (4.22) and (4.26) for  $U = 1, \alpha = 0.2, \rho = 45.385, \nu = 0.2288$  and different values of  $t$ .

## 6 Analysis and Numerical Results:

In order to discuss some significant physical aspects of the obtained results; several graphs are sketched in this section. Multiple numbers of diagrams of the velocity  $u(y, t)$  and the shear stress  $\tau(y, t)$  against " $y$ " are performed for diverse situations of representative values. For instance, we choose  $U = 1, \rho = 45.385, \nu = 0.2288$  and  $\alpha = 0.2$  for simplicity and different values of  $t, \alpha, \nu$  and  $y$  are elected to illustrate their effects on fluid motion.

From Fig. 4.2, it is apparent that the velocity is an increasing function with respect to " $t$ " whereas shear stress in magnitude is also increasing with regard to " $t$ ". It is also clear from Fig. 4.2(a), boundary condition is being satisfied.

Fig. 4.3 shows the disparity of two physical entities with respect to rheological parameter " $\alpha$ ". As it was to be expected both the velocity and shear stress (of course in absolute sense) are increasing functions with respect to " $\alpha$ ".

The impact of kinematic viscosity  $\nu$  is emphasized by Fig. 4.4, which shows that velocity as well as shear stress increases (in magnitude) with regard to  $\nu$ .

Furthermore to see the effect of moving plane on the fluid motion, Fig. 4.5 is depicted against time  $t$  for different values of  $y$ . It is clear from these figures that both velocity and shear stress are decreasing functions of  $y$ .

At last for the comparison of the velocity field and shear stress corresponding to the two models, i.e. Newtonian and second grade fluid are collectively depicted in Fig. 4.6, for three different values of time and material constant. It is clearly seen from these figures that second grade fluid and Newtonian fluid exhibit almost the same behavior near the plate. As we move far away from the plate, the second grade fluid comparatively becomes faster than the Newtonian fluid. However, the shear stress of the second grade is larger than the Newtonian fluid all over the domain. It is also understandable from these figures that the non-Newtonian effects vanish in time i.e. for large time  $t$ . Thus the motion of the second grade fluid can be estimated by the behavior of Newtonian fluid.

## 7 Concluding Remarks:

In this section, the velocity field  $u(y, t)$  and the adequate shear stress  $\tau(y, t)$  corresponding to the flow of second grade fluid when the plane is moving with constant acceleration are determined by using Fourier sine and Laplace transforms. The results that have been obtained are presented under integral form in terms of elementary functions  $\exp(\cdot)$ ,  $\sin(\cdot)$ ,  $\cos(\cdot)$  and their respective angles and satisfy all imposed initial and boundary conditions. They are written as the sum of large time and transient solution and can be simply reduced to the similar solution for Newtonian fluid. Finally in order to bring light on some physical aspects of the obtained results; the influence of

material parameters on the fluid motion is manifested by graphical illustrations. A comparison between Newtonian and second grade fluid is also realized. The main outcomes of this study are as follows:

1. The general solution (4.15) and (4.21) for second grade fluid are presented as the sum of large time and transient solution. They have been instantly particularized to give the similar solutions of Newtonian fluid.
2. The velocity field  $u(y, t)$  and the adequate shear stress  $\tau(y, t)$  (in absolute vale) are both increasing functions respectively with respect to time " $t$ ".
3. The material parameter " $\alpha$ " have analogous effects on the fluid motion both velocity and shear stress are increasing functions with respect to " $\alpha$ ".
4. The velocity field and shear stress both increases with the increasing value of kinematic viscosity " $\nu$ ".
5. The velocity field and shear stress tends to zero as we move away from the plate.
6. The second grade fluid is comparatively swifter than the Newtonian fluid as we move away from the plate.
7. The non-Newtonian effects vanish in time.



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